Quantum Morphisms

Lecture 12

Last Week

$C(Qut(G))$ – the universal C^*-algebra generated by u_{ij} for $i,j \in V(G)$ satisfying:

$$\left.\begin{array}{l} 1)\ u_{ij} = u_{ij}^2 = u_{ij}^* \quad \forall\, i,j \in V(G) \\ 2)\ \sum_k u_{ik} = 1 = \sum_\ell u_{\ell j} \quad \forall\, i,j \in V(G) \end{array}\right\} \mathcal{U} = (u_{ij}) \text{ is a QPM}$$

3) $A_G \mathcal{U} = \mathcal{U} A_G$

- $Qut(G) = (C(Qut(G)), \mathcal{U})$ is the quantum automorphism group of G.
- $\mathcal{U}$ – the fundamental representation of $Qut(G)$.
- Universal property of $C(Qut(G))$: if $\mathcal{P} = (p_{ij}) \in M_n(\mathcal{A})$ is a QPM s.t. $A_G \mathcal{P} = \mathcal{P} A_G$, then $\exists$ a $*$-hom $\phi : C(Qut(G)) \to \mathcal{A}$ s.t. $\phi(u_{ij}) = p_{ij}$.

Properties of $Qut(G)$

Comultiplication: $\Delta : C(Qut(G)) \to C(Qut(G)) \otimes C(Qut(G))$

$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ is a $*$-hom

Antipode: $S : C(Qut(G)) \to C(Qut(G))^{op}$ $\quad S(ab) = S(b)S(a)$

$S(u_{ij}) = u_{ji}$ is a $*$-hom

Counit: $\varepsilon : C(Qut(G)) \to \mathbb{C}$

$$\varepsilon(u_{ij}) = \delta_{ij} \quad \text{is a } *\text{-hom}$$

Haar state: $h : C(Qut(G)) \to \mathbb{C}$ satisfying

$$(h \otimes id) \circ \Delta = (id \otimes h) \circ \Delta = h$$

For $Qut(G)$, h is tracial.

Intertwiners

$\mathcal{U}^{\otimes k}$ - $V(G)^k \times V(G)^k$ matrix with

$$(\mathcal{U}^{\otimes k})_{i_1 \cdots i_k, j_1 \cdots j_k} = u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_k j_k}$$

$$\mathcal{U}^{\otimes 0} = (1)$$

An (ℓ, k)-intertwiner of $Qut(G)$ is a matrix $T \in \mathbb{C}^{V(G)^\ell \times V(G)^k}$ s.t. $\mathcal{U}^{\otimes \ell} T = T \mathcal{U}^{\otimes k}$.

$C_q^G(\ell, k) :=$ set of (ℓ, k)-intertwiners of $Qut(G)$.

$C_q^G := \bigcup_{\ell, k} C_q(\ell, k)$

C_q^G is a tensor category with duals.

Theorem (Chassaniol):

$$C_q^G = \langle M^{1,2}, M^{1,0}, A_G \rangle_{+,\circ,\otimes,*}$$
$$C^G = \langle M^{1,2}, M^{1,0}, A_G, S \rangle_{+,\circ,\otimes,*}$$

$M^{1,2}(e_i \otimes e_j) = \delta_{ij} e_i$ $\qquad$ $M^{1,0}$ = all 1's vector

$S(e_i \otimes e_j) = e_j \otimes e_i$

$C^G(\ell, k)$ = span of characteristic matrices of orbits of $Aut(G)$ on $V(G)^\ell \times V(G)^k$.

$M \in C^G(\ell,k) \Leftrightarrow P^{\otimes \ell} M = M P^{\otimes k}$ $\forall$ $P \in Aut(G)$.

$M \in C_q^G(\ell,k) \Leftrightarrow \mathcal{P}^{\otimes \ell} M = M \mathcal{P}^{\otimes k}$ for any QPM $\mathcal{P}$ that commutes w/ A_G.

Today (Lupini, Mančinska, Roberson)

We will define orbits and orbitals (orbits on $V(G) \times V(G)$) of $Qut(G)$, and show that $C_q^G(1,0)$ & $C_q^G(1,1)$ are the span of the characteristic vectors/matrices of these orbits & orbitals. We will also characterize quantum isomorphism of G & H in terms of $Qut(G \cup H)$.

Orbits & Orbitals of Qut(G)

Let $\mathcal{U}=(u_{ij})$ be the fundamental representation of $Qut(G)$.

We define the following relations on $V(G)$ & $V(G)\times V(G)$:

1) $x\sim_1 y$ if $u_{xy}\neq 0$

2) $(x,x')\sim_2(y,y')$ if $u_{xy}u_{x'y'}\neq 0$

Classically: $u_{xy}\neq 0 \Leftrightarrow \exists\, P\in Aut(G)$ s.t. $P_{xy}=1$

$u_{xy}u_{x'y'}\neq 0 \Leftrightarrow \exists\, P\in Aut(G)$ s.t. $P_{xy}P_{x'y'}=1$

Lemma: Both $\sim_1$ & $\sim_2$ are equivalence relations.

Proof: We prove it for $\sim_1$.

Reflexivity: Recall the counit $\varepsilon: C(Qut(G))\to\mathbb{C}$ is a $*$-homomorphism and $\varepsilon(u_{xy})=\delta_{xy}$. Thus

$$\varepsilon(u_{xx})=1 \Rightarrow u_{xx}\neq 0, \quad \text{i.e. } x\sim_1 x.$$

Symmetry: Use the antipode $S: C(Qut(G))\to C(Qut(G))^{op}$.

Suppose that $x\sim_1 y$, i.e. $u_{xy}\neq 0$. Then

$$S(u_{yx})=u_{xy}\neq 0 \Rightarrow u_{yx}\neq 0, \quad \text{i.e. } y\sim_1 x.$$

Transitivity: Use comultiplication $\Delta: C(Qut(G)) \to C(Qut(G))^{\otimes 2}$.

Suppose that $x \sim_1 y$ & $y \sim_1 z$, i.e. $u_{xy} \neq 0$ & $u_{yz} \neq 0$. Then

$$
\begin{aligned}
(u_{xy} \otimes u_{yz}) \Delta(u_{xz}) &= (u_{xy} \otimes u_{yz}) \sum_w u_{xw} \otimes u_{wz} \\
&= \sum_w u_{xy} u_{xw} \otimes u_{yz} u_{wz} \\
&= u_{xy} \otimes u_{yz} \neq 0
\end{aligned}
$$

Thus $u_{xz} \neq 0$, i.e. $x \sim_1 z$.

So we have shown that $\sim_1$ is an equivalence relation.

Proof for $\sim_2$ left as exercise. $\square$

We now define the orbits and orbitals of $Qut(G)$ as the equivalence classes of $\sim_1$ & $\sim_2$ respectively.

Note: $x \sim_1 y \Leftrightarrow u_{xy} \neq 0 \Leftrightarrow u_{xy} u_{xy} \neq 0 \Leftrightarrow (x,x) \sim_2 (y,y)$, and $(x,x) \nsim_2 (y,z)$ if $y \neq z$ since $u_{xy} u_{xz} = 0$ in this case. Thus the orbits are the orbitals that are contained in the diagonal of $V(G) \times V(G)$.

Coherent Configurations/Algebras

A coherent configuration on a set X is a partition $\mathcal{R} = \{R_i : i \in \mathcal{I}\}$ of $X \times X$ into the relations/classes R_i satisfying the following:

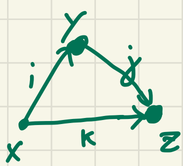
1) there is a subset $\mathcal{D} \subseteq \mathcal{I}$ s.t. $\{R_d : d \in \mathcal{D}\}$ is a partition of the diagonal $\{(x,x) : x \in X\}$;

2) for each R_i, its converse/transpose $\{(y,x) : (x,y) \in R_i\}$ is a relation, say $R_{i'}$, in $\mathcal{R}$ ($R_{i'} = R_i$ is possible);

3) $\forall\ i,j,k \in \mathcal{I}$ there exists $p_{ij}^k \in \mathbb{N}$ such that for any $(x,z) \in R_k$:

intersection numbers

$$\left|\{y \in X : (x,y) \in R_i \ \&\ (y,z) \in R_j\}\right| = p_{ij}^k .$$



$$A^i A^j = \sum_k p_{ij}^k A^k$$

Characteristic matrix of R_i:

$$(A^i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{o.w.} \end{cases}$$

The span of the matrices A^i is a coherent algebra, i.e. a subalgebra $\mathcal{A}$ of $\mathbb{C}^{X \times X}$ s.t.

1) $I, J \in \mathcal{A}$;
2) $M \in \mathcal{A} \Rightarrow M^* \in \mathcal{A}$ ($\mathcal{A}$ is self-adjoint);
3) $\mathcal{A}$ is closed under Schur/entrywise product.

Conversely, any coherent algebra will have a unique basis of orthogonal (wrt $\langle A, B \rangle = Tr(AB^*) = sum(A \circ \overline{B})$) 01-matrices (minimal Schur idempotents) and these will be the characteristic matrices of some coherent configuration.

Remark: $span\{A^i : i \in \mathcal{I}\}$ = matrices constant on the classes of $\mathcal{R}$.

i.e.

$M \in \mathcal{A} \Leftrightarrow M_{xx'} = M_{yy'}$ whenever $\exists\, i \in \mathcal{I}$ s.t. $(x,x'), (y,y') \in R_i$

Examples of coherent configurations/algebras

- 2-class scheme: $R_1 = \{(x,x) : x \in X\}$, $R_2 = \{(x,y) : x \neq y\}$

  $A^1 = I$ $\qquad\qquad A^2 = J - I$

- Singleton partition: $\{(x,y)\} \in \mathcal{R}$ $\forall\, x,y \in X$.

  Char. matrices: $E_{ij} = e_i e_j^*$.

- Strongly Regular Graphs

  G regular, any pair of adjacent vts share λ neighbors

  any pair of distinct non-adj vts share μ neighbors

  Then $\operatorname{span}\{I, A_G, A_{\bar{G}} = J - I - A_G\}$ is a coherent algebra.

  $R_1 = \{(x,x) : x \in V(G)\}$ $\quad R_2 = E(G)$ $\quad R_3 = E(\bar{G})$

Coherent Configurations/Algebras Associated to Graphs

Note: The intersection of two coh. algs. is a coh. alg.

The coherent algebra (configuration) of a graph G is the smallest coh. alg. containing A_G.

This can be computed efficiently by the Weisfeiler-Leman algorithm.

The orbital configuration of G is the partition of $V(G) \times V(G)$ into the orbitals of $\mathrm{Aut}(G)$.
The corresponding orbital algebra of G is the commutant of $\mathrm{Aut}(G)$:

$$\{ M \in \mathbb{C}^{V(G) \times V(G)} : MP = PM \text{ for all } P \in \mathrm{Aut}(G) \}$$

The Quantum Orbital Algebra

Lemma: Let $\mathcal{U}=(u_{xy})$ be an $X\times Y$ quantum perm. mtx. For $M\in\mathbb{C}^{X\times X}$ + $N\in\mathbb{C}^{Y\times Y}$,

$$M\mathcal{U}=\mathcal{U}N \iff M_{xx'}=N_{yy'} \text{ whenever } u_{xy}u_{x'y'}\neq 0$$

Proof: Let $x,x'\in X$ + $y,y'\in Y$. Consider

$$u_{xy}(M\mathcal{U})_{xy'}u_{x'y'}=u_{xy}\Big(\sum_{x''}M_{xx''}u_{x''y'}\Big)u_{x'y'}=M_{xx'}u_{xy}u_{x'y'}$$

$$u_{xy}(\mathcal{U}N)_{xy'}u_{x'y'}=u_{xy}\Big(\sum_{y''}u_{xy''}N_{y''y'}\Big)u_{x'y'}=N_{yy'}u_{xy}u_{x'y'}$$

Thus $M\mathcal{U}=\mathcal{U}N \Rightarrow M_{xx'}u_{xy}u_{x'y'}=N_{yy'}u_{xy}u_{x'y'}$

$$\Rightarrow M_{xx'}=N_{yy'} \text{ if } u_{xy}u_{x'y'}\neq 0.$$

Conversely, if $M_{xx'}=N_{yy'}$ whenever $u_{xy}u_{x'y'}\neq 0$, then

$$\underbrace{\sum_{x'y}u_{xy}(M\mathcal{U})_{xy'}u_{x'y'}}_{=(M\mathcal{U})_{xy'}}=\underbrace{\sum_{x'y}u_{xy}(\mathcal{U}N)_{xy'}u_{x'y'}}_{=(\mathcal{U}N)_{xy'}}$$

Corollary: Let $\mathcal{U}$ be the fundamental representation of $Qut(G)$. Then $M\mathcal{U} = \mathcal{U}M$ (i.e, M is a (1,1)-intertwiner of $Qut(G)$) if & only if M is constant on the orbitals of $Qut(G)$.

Corollary: The orbitals of $Qut(G)$ form a coherent configuration, i.e. $C_q^G(1,1)$ is a coherent algebra.

Proof: $C_q^G(1,1)$ is an algebra and $I, J \in C_q^G(1,1)$ is trivial.

If $M\mathcal{U} = \mathcal{U}M$, then $\mathcal{U}^* M^* = M^* \mathcal{U}^*$ and thus

$$\begin{array}{ccc} \mathcal{U}(\mathcal{U}^* M^*)\mathcal{U} & = & \mathcal{U}(M^* \mathcal{U}^*)\mathcal{U} \\ \| & & \| \\ M^* \mathcal{U} & & \mathcal{U} M^* \end{array}$$

Thus $C_q^G(1,1)$ is self-adjoint.

Lastly, the previous corollary shows that $C_q^G(1,1)$ is closed under Schur product. $\square$

Alternative proof: Show that $M^{1,2}(A \circ B)(M^{1,2})^* = A \bullet B$.

Theorem: $\mathcal{U}\psi = \psi \mathcal{U}^{\otimes 0} \Leftrightarrow \psi$ is constant on the orbits of $Qut(G)$.

Also the orbits form an equitable partition.

Corollary: Let $R_1, \dots, R_k$ be the classes of the coherent configuration of G, and $A^1, \dots, A^k$ their characteristic matrices. Let $\mathcal{U} = (u_{xy})$ be the fundamental representation of $Qut(G)$. Then $u_{xy} u_{x'y'} = 0$ if (x,x') + (y,y') are not contained in some common class R_i. In other words, $A^i \mathcal{U} = \mathcal{U} A^i$ for all $i = 1, \dots, k$.

Proposition (Babai + Kucera?): Almost all graphs have their coherent algebra equal to the full matrix algebra.

Corollary: Almost all graphs have trivial quantum automorphism group.

Theorem (Junk, Schmidt, Weber): Almost all trees have quantum symmetry, i.e. $C(Qut(G))$ is noncommutative.

The Haar State

Lemma: Let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be the orbits of $Qut(G)$ and let $R_1, \ldots, R_s$ be its orbitals. If $h: C(Qut(G)) \to \mathbb{C}$ is the Haar state of $Qut(G)$, then

$$h(u_{xy}) = \begin{cases} |\mathcal{O}_i|^{-1} & \text{if } x, y \in \mathcal{O}_i \\ 0 & \text{o.w.} \end{cases}$$

$$h(u_{xy} u_{x'y'}) = \begin{cases} |R_i|^{-1} & \text{if } (x, x'), (y, y') \in R_i \\ 0 & \text{o.w.} \end{cases}$$

Proof: Exercise.

Note: $u_{xy} \neq 0 \Rightarrow h(u_{xy}) \neq 0$.

Theorem 1: Let G & H be graphs. Then $G \cong_{qc} H$ if & only if there is a QPM $\mathcal{P}$ s.t. $A_G \mathcal{P} = \mathcal{P} A_H$.

Theorem 2: Let G & H be connected graphs. Then $G \cong_{qc} H$ if & only if $\exists\, g \in V(G)$ & $h \in V(H)$ in the same orbit of $Qut(G \cup H)$.

$J-I-A_G \quad J-I-A_H$

Proof of 1: Suppose that $\mathcal{P}$ is a QPM s.t. $A_G \mathcal{P} = \mathcal{P} A_H$. By taking complements of both G & H, we may assume that G is connected. It follows that H is connected (Exercise). Consider $A_H \mathcal{P}^* = (\mathcal{P} A_H)^* = (A_G \mathcal{P})^* = \mathcal{P}^* A_G$

$$\begin{pmatrix} A_G & 0 \\ 0 & A_H \end{pmatrix}\begin{pmatrix} 0 & \mathcal{P} \\ \mathcal{P}^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_G \mathcal{P} \\ A_H \mathcal{P}^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{P} A_H \\ \mathcal{P}^* A_G & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \mathcal{P} \\ \mathcal{P}^* & 0 \end{pmatrix}\begin{pmatrix} A_G & 0 \\ 0 & A_H \end{pmatrix}$$

i.e. $\begin{pmatrix} 0 & \mathcal{P} \\ \mathcal{P}^* & 0 \end{pmatrix}$ commutes with $A_{G \cup H}$.
(and is a QPM)

Now let $\mathcal{U}$ be the fundamental representation of $Qut(G \cup H)$. By the above $\exists\, g \in V(G)$ & $h \in V(H)$ s.t. $u_{gh} \neq 0$.

Note that for any $g,g',g''\in V(G)$ & $h\in V(H)$, the pairs (g,g') & (h,g'') are in different orbitals of $Qut(G\cup H)$. Thus $u_{gh}u_{g'g''}=0$. Similarly, $u_{gh}u_{h'h''}=0$ for any $g\in V(G)$ & $h,h',h''\in V(H)$.

$$\mathcal{U} = \begin{array}{cc} & \begin{array}{cc} V(G) & V(H) \end{array} \\ \begin{array}{c} V(G) \\ V(H) \end{array} & \left(\begin{array}{c|c} u_{g'g''} & u_{gh} \\ \hline & u_{h'h''} \end{array}\right) \end{array} = \begin{pmatrix} * & \hat{\mathcal{U}} \\ * & * \end{pmatrix}$$

For each $g\in V(G)$ define $u_g=\sum_{h\in V(H)} u_{gh}$. Similarly define u_h for $h\in V(H)$. We aim to show that u_g/u_h does not depend on g/h & $u_g=u_h$. I.e. want to show $\hat{\mathcal{U}}$ is a sub-QPM. We have

$$0=u_g\Big(\sum_{g''\in V(G)} u_{g'g''}\Big)=u_g(1-u_{g'})=u_g-u_gu_{g'}$$

$\Rightarrow$ $u_g=u_gu_{g'}$ $\forall g'\in V(G)$. Similarly, $u_g=u_{g'}u_g$ $\forall g'\in V(G)$.

Thus $u_g=u_gu_{g'}=u_{g'}$ $\forall g,g'\in V(G)$. Similarly $u_h=u_{h'}$.

Now $|V(G)|\,u_g=\sum_{\substack{g'\in V(G)\\ h'\in V(H)}} u_{g'h'}=|V(H)|\,u_h \Rightarrow u_g=u_h$.

Denote this non-zero projection $\hat{1}$, and let $\hat{\mathcal{A}}$ be the

C^*-algebra generated by the entries of $\hat{\mathcal{U}}$. Note $\hat{1} u_{gh} = \left(\sum_{h'} u_{gh'}\right) u_{gh} = u_{gh}$ + similarly $u_{gh}\hat{1} = u_{gh}$.

Thus $\hat{1}$ is the identity in $\hat{\mathcal{A}}$ and so $\hat{\mathcal{U}}$ is a QPM over $\hat{\mathcal{A}}$. Moreover, $A_{G \cup H}\mathcal{U} = \mathcal{U}A_{G\cup H}$ implies

$$\begin{pmatrix} A_G & 0 \\ 0 & A_H \end{pmatrix}\begin{pmatrix} * & \hat{\mathcal{U}} \\ * & * \end{pmatrix} = \begin{pmatrix} * & \hat{\mathcal{U}} \\ * & * \end{pmatrix}\begin{pmatrix} A_G & 0 \\ 0 & A_H \end{pmatrix}$$

$$\begin{array}{ccc} \| & & \| \\ \begin{pmatrix} * & A_G\hat{\mathcal{U}} \\ * & * \end{pmatrix} & & \begin{pmatrix} * & \hat{\mathcal{U}}A_H \\ * & * \end{pmatrix} \end{array}$$

Thus $A_G\hat{\mathcal{U}} = \hat{\mathcal{U}}A_H$. Lastly, $h(\hat{1}) \neq 0$ $(= \frac{1}{2}$ in fact$)$ since there is some $u_{gh} \neq 0$. Let $\hat{h} = h(\hat{1})^{-1} h|_{\hat{\mathcal{A}}}$.

Then $\hat{h}(\hat{1}) = 1$ and thus $\hat{h}$ is a tracial state on $\hat{\mathcal{A}}$.

Therefore $\hat{\mathcal{U}}$ is a QPM over a C^*-algebra that has a tracial state s.t. $A_G\hat{\mathcal{U}} = \hat{\mathcal{U}}A_H$, i.e. $G \cong_{qc} H$. $\square$

Brannan, Chirvasitu, Eifler, Harris, Paulsen, Su, Wasilewski:

$G \cong_{qc} H$ if & only if there is a QPM $\mathcal{P}$ over a $*$-algebra such that $A_G \mathcal{P} = \mathcal{P} A_H$.

QPM over $*$-algebra: same as over C^*-algebra except we must explicitly require that

$$p_{ij} p_{ik} = 0 \text{ if } j \neq k, \text{ and } p_{ij} p_{\ell j} = 0 \text{ if } i \neq \ell.$$

Isomorphisms of Coherent Algebras

Suppose that $\mathcal{A}$ & $\mathcal{A}'$ are coherent algebras.

A (weak) isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ is a bijective linear map $\phi : \mathcal{A} \to \mathcal{A}'$ such that

1) $\phi(AB) = \phi(A)\phi(B)$, i.e. ϕ is an algebra isomorphism;
2) $\phi(A \circ B) = \phi(A) \circ \phi(B)$;
3) $\phi(A^*) = \phi(A)^*$;
4) $\phi(I) = I$ & $\phi(J) = J$.

If p_{ij}^k for $i,j,k \in \mathcal{I}$ and q_{ij}^k for $i,j,k \in \mathcal{I}'$ are the intersection numbers of $\mathcal{A}$ & $\mathcal{A}'$ respectively, then the existence of an isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ is equivalent to the existence of a bijection $\pi : \mathcal{I} \to \mathcal{I}'$ s.t. $p_{ij}^k = q_{\pi(i)\pi(j)}^{\pi(k)}$ $\forall\, i,j,k \in \mathcal{I}$, and the corresponding isomorphism $\phi : \mathcal{A} \to \mathcal{A}'$ is given by $\phi(A^i) = A'^{\pi(i)}$.

Theorem: Let $\mathcal{A}_G$ & $\mathcal{A}_H$ be the quantum orbital algebras of G & H respectively. If $G \cong_{qc} H$, then there is an isomorphism $\phi: \mathcal{A}_G \to \mathcal{A}_H$ such that $\phi(A_G) = \phi(A_H)$.

Corollary: Let $\mathcal{A}_G$ & $\mathcal{A}_H$ be the coherent algebras of G & H respectively. If $G \cong_{qc} H$, then there is an isomorphism $\phi: \mathcal{A}_G \to \mathcal{A}_H$ s.t. $\phi(A_G) = \phi(A_H)$.

Corollary: If $G \cong_{qc} H$, then G and H are <u>not</u> distinguishable by the (2-dimensional) Weisfeiler-Leman algorithm.

Homework: Let $M^{2,1} = (M^{1,2})^*$. Then the matrix $M^{1,2}\left((A_G M^{1,2}) \otimes I\right) A_G^{\otimes 3} \left(I \otimes (M^{2,1} A_G)\right) M^{2,1}$ is in $C_q^G(1,1)$. Is it necessarily in the coherent algebra of G? What are its entries?